Possibility of automatic Pauli-Villars-like regularization through extra dimensions

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Abstract

Fermions in a space with some extra dimensional reflection symmetries are considered. In this space only one Kaluza-Klein mode is observed at the scales larger than the sizes of the extra dimensions while at the smaller scales all modes become observable. The resulting picture suggests that this model has a built-in Pauli-Villars-like regularization as its inherent characteristic.

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I. INTRODUCTION

Extra dimensions are attractive frameworks to address many problems in high energy physics. They have a good prospect to account for many theoretical and phenomenological problems that the standard model can not answer, such as the hierarchy between the strengths of gravitational and electroweak interactions, fermion masses, families, chirality, cosmological constant problem, Higgs-Gauge unification etc. in addition to being the standard setting for string theory. The standard practice for extra dimensions is to take them be compact (at least at the energies much smaller than the Planck scale). This, in turn, results in an infinite number of Fourier modes for an extra dimensional field that are called Kaluza-Klein (KK) modes of that field [1, 2]. Although extra dimensional models are promising candidates for the physics beyond the standard model these models have a technically unpleasing aspect; the infinite number of KK modes are additional sources for infinities in the corresponding quantum field theories. Kaluza-Klein modes make the issue of the regularization more intricate even when one lets extra dimensional models be effective field theories [3, 4]. In this paper I introduce a model where the Kaluza-Klein modes serve for regularization on contrary to the generic case where Kaluza-Klein modes make the regularization more difficult.

In some of my recent studies I had considered metric reversal symmetry as a possible cure to cosmological constant and zero point energy problems of quantum fields [5, 6], and in [7] I had considered fermions and a variant of the metric reversal symmetry to get a finite number of Kaluza-Klein (KK) modes and their ghosts at the scales larger than the size of extra dimensions while all KK modes are observed at smaller scales. In this study I consider fermions and use the same symmetry used in [7] to construct a space where only one KK mode is observed at the scales larger than the size of the extra dimensions while a usual KK mode and its ghost are observed at smaller length scales. The resulting picture amounts to an automatic built-in Pauli-Villars [8] like regularization. The main elements of this scheme are two extra dimensional discrete symmetries in conjunction with non-trivial boundary conditions imposing specific forms for the Lagrangians at different energy scales. The details of the scheme are given in the following sections. In the next section the space, the symmetries, and the boundary conditions are specified. In the remaining sections the scheme is introduced and studied.

II. THE SETTING, THE SYMMETRIES, AND THE BOUNDARY CONDITIONS OF THE MODEL

Consider the following 7-dimensional space

$$ds^2 \;=\; g_{\mu\nu}(x)\, dx^\mu dx^\nu \;-\; \cos^2 k_2 y_2\, [\, dy_1^2 \,+\, \cos^2 k_3 y_3 dy_2^2 \,+\, dy_3^2\,] \qquad \mu,\nu \,=\, 0,1,2,3 \quad (1)$$

In fact seven is the minimum number of dimensions that can be taken in this scheme. This point will be discussed at the end of the paragraph after Eq.(28). I take the extra dimensions be compact and have the sizes L_1, L_2, L_3, and $k_1 = \frac{2\pi}{L_1}$, $k_2 = \frac{2\pi}{L_2}$, $k_3 = \frac{2\pi}{L_3}$. The action for matter fields in this space is

$$S_f \;=\; \int \mathcal{L}_f\, \cos^3 k_2 y_2\, \cos k_3 y_3\, d^4x\, dy_1\, dy_2\, dy_3 \qquad (2)$$

where $\mathcal{L}_f$ denotes the Lagrangian corresponding to matter fields. Note that the extra dimensional contribution to the Einstein-Hilbert action for the metric (1) vanishes after integration over y_2 and y_3. In other words (1) is effectively equivalent to its 4-dimensional part at the scales much larger than L_2 and L_3. So one does not need to bother with energy-momentum tensor content necessary to support the extra dimensional piece of (1) at current accessible scales.

I consider the following transformations

$$x^a \;\rightarrow\; -\,x^a \quad , \qquad a = 0,1,2,3,5 \qquad (3)$$

$$x^b \;\rightarrow\; -\,x^b \quad , \qquad b = 0,1,2,3,6 \qquad (4)$$

where $x^5 = y_1$, $x^6 = y_2$. The general Fourier decomposition of a field φ in the coordinate z in the presence of the symmetry (3) or (4) in either of the 5th or 6th directions in space may be expressed as

$$\begin{aligned}
\varphi(x,z) \;&=\; \sum_{n=-\infty}^{\infty} [\, \alpha_n(x)\, \sin(\frac{1}{2} n\, kz) \,+\, \beta_n(x)\, \cos(\frac{1}{2} n\, kz)\,] \\
&=\; \sum_{n=0}^{\infty} \varphi_{|n|}(x)\, \sin(\frac{1}{2}|n|\, kz) \,+\, \tilde{\varphi}_{|n|}(x)\, \cos(\frac{1}{2}|n|\, kz) \\
&=\; \sum_{n=0}^{\infty} [\, a_{|n|}\, \sin(\frac{1}{2}|n|\, kz) \,+\, b_{|n|}\, \cos(\frac{1}{2}|n|\, kz)\,]\, \varphi_{|n|}(x) \\
&=\; \sum_{n=0}^{\infty} \{\, f_{|n|}[\cos(\frac{|n|kz}{2}) + \sin(\frac{|n|kz)}{2})] \,+\, g_{|n|}[\cos(\frac{|n|kz)}{2}) - \sin(\frac{|n|kz)}{2})]\,\}\varphi_{|n|}(x)
\end{aligned}$$

$$
\begin{aligned}
&\varphi_{|n|}(x) = \alpha_{|n|}(x) - \alpha_{-|n|}(x) \quad , \quad \tilde{\varphi}_{|n|}(x) = \beta_{|n|}(x) + \beta_{-|n|}(x) \\
&f_{|n|} = \frac{1}{2}(b_{|n|} + a_{|n|}) \quad , \quad g_{|n|} = \frac{1}{2}(b_{|n|} - a_{|n|}) \quad , \quad a_{|n|}^2 + b_{|n|}^2 = 1 \\
&z = y_1, y_2 \quad , \quad k = k_1, k_2
\end{aligned} \tag{5}
$$

where $a_{|n|}$, $b_{|n|}$, $f_{|n|}$, $g_{|n|}$ are some constants. The absolute value signs enclosing n in (5) are employed only to emphasize that those n's are positive integers. Even and odd n correspond to periodic and anti-periodic boundary conditions [11], respectively. The proceeding from the second line of (5) to the third one follows from the fact that all Kaluza-Klein modes are the same except their masses, and the sine and cosine terms in (5) for the same n result in the same mass so they correspond to the same physical field. An expansion similar to (5) is also true for the 7th direction, y_3 while in that case passing from the first line to the second line of the equation does not hold since there is no symmetry similar to (3) or (4) for the 7th direction. So both positive and negative values of n should be included in (5) for the expansion corresponding to $z = y_3$. The 4-dimensional parts of the Kaluza-Klein modes are taken to transform, under (3) and (4), as

$$
\begin{aligned}
&\varphi_{n,m,r}(x) \rightarrow \xi^{\lambda_n} \mathcal{CPT}\, \varphi_{n,m,r}(-x) \quad \text{as} \quad x^a \rightarrow -x^a && (6)\\
&\varphi_{n,m,r}(x) \rightarrow \xi^{\lambda_m} \mathcal{CPT}\, \varphi_{n,m,r}(-x) \quad \text{as} \quad x^b \rightarrow -x^b && (7)\\
&\varphi_{n,m,r}(x) \rightarrow \xi^{\lambda_n+\lambda_m} \mathcal{CPT}\, \varphi_{n,m,r}(x) \quad \text{as} \quad x^a \rightarrow -x^a \quad , \quad x^b \rightarrow -x^b && (8)\\
&\lambda_n = \frac{i}{2}(-1)^{\frac{n}{2}} \quad \lambda_m = \frac{i}{2}(-1)^{\frac{m}{2}} \qquad a = 0,1,2,3,5 \;\; ; \;\; b = 0,1,2,3,6
\end{aligned}
$$

where n, m, r are the modes corresponding to y_1, y_2, y_3 directions, respectively; ξ is some constant other than 1 or -1, and $\mathcal{CPT}$ denotes the part of (4-dimensional) CPT transformation acting on the spinor part of the field. Here I take the extra dimensional reflections essentially act only on the positions of the fields while they do not act on the spinor parts of the fields. So it is more natural to take $\mathcal{CPT}$ rather than $\mathcal{PT}$ since $\mathcal{CPT} \propto \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and commutes with extra dimensional gamma matrices while $\mathcal{PT} \propto \gamma_5\gamma^2$ and does not commute with the extra dimensional gamma matrices [9, 10]. The natural choice for ξ would be 1 or -1 if no restriction is imposed on the couplings of different Kaluza-Klein modes. On the contrary I want to impose coupling of specified modes with each other e.g. excluding diagonal coupling of the modes while imposing the coupling of $n = 4k + 1$ modes to $n = 4k + 3$ modes in (20) below. This imposition naturally follows when the transformations (3), (4)

are supplemented by (6), (7): The volume element in (2) is odd under either of (3) or (4) while the volume element plus the integration boundaries is even under either of these transformations. This requires the 4-dimensional part of the kinetic term in $\mathcal{L}_f$ be even under either of the transformations. The 4-dimensional kinetic term of $\mathcal{L}_f$ is invariant under the 4-dimensional CPT transformation. This guarantees the 4-dimensional kinetic term in $\mathcal{L}_f$ be even under extra dimensional part of the transformations in (3), (6) (or (4), (7)). This, in turn, requires $n = 4k+1$ modes couple to $n = 4k+3$ modes in (20). The significance of this type of imposition on the form of the Lagrangian will be evident when we consider the resulting field spectrum in the following paragraphs. So ξ is taken to be an arbitrary constant other than 1 or -1. It is evident that the possible values of $\lambda_{n(m)}$ are $\frac{i}{2}$, $-\frac{i}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$. Hence the terms of the form $\bar{\psi}_{n_1,n_2}\psi_{m_1,m_2}$ are invariant under (6) and/or (7) only for specific values of n and m. This point as well will be used to construct correct action functionals in the following part of the paper. I also give the transformation rule of the fields under the simultaneous application of (6) and (7) as a reference for the evaluation of S_{fk2} given in (28)

$$\begin{aligned}
&\varphi_{n,m,r}(x) \;\rightarrow\; \xi^{\mp 1}\mathcal{CPT}\,\varphi_{n,m,r}(-x) \quad \text{if} \quad n = 4l+1\;,\; m = 4p+1 \\
&\varphi_{n,m,r}(x) \;\rightarrow\; \mathcal{CPT}\,\varphi_{n,m,r}(-x) \quad \text{if} \quad n = 4l+1\;,\; m = 4p+3 \quad \text{or} \quad n = 4l+3\;,\; m = 4p+1 \\
&\varphi_{n,m,r}(x) \;\rightarrow\; \xi^{\pm 1}\mathcal{CPT}\,\varphi_{n,m,r}(-x) \quad \text{if} \quad n = 4l+3\;,\; m = 4p+3 \\
&l,p = 0,1,2,.....
\end{aligned} \tag{9}$$

I adopt anti-periodic boundary conditions [11] for the 5th and 6th directions while I take periodic boundary conditions for the 7th direction. So n's in (5) are odd integers for the 5th and 6th directions, y_1 and y_2 while they are even integers for the 7th direction, y_3. The reason for adopting anti-periodic boundary conditions for y_1 and y_2, and periodic conditions for y_3 will be discussed in the paragraph after Eq.(12). I also introduce the following symmetry transformations

$$k_1 y_1 \;\rightarrow\; k_1 y_1 \,+\, \pi \tag{10}$$

$$k_2 y_2 \;\rightarrow\; k_2 y_2 \,+\, \pi \tag{11}$$

One observes that

$$\begin{aligned}
&\text{as} \quad ky \rightarrow ky + \pi \\
&i)\ \text{if} \;\; n = 4l+1 \;\Rightarrow\; (\cos\frac{n}{2}ky \,+\, \sin\frac{n}{2}ky\,) \rightarrow (\cos\frac{n}{2}ky \,-\, \sin\frac{n}{2}ky\,)
\end{aligned}$$

$$
\begin{aligned}
&(\cos\frac{n}{2}ky \,-\, \sin\frac{n}{2}ky\,) \,\rightarrow\, -(\,\cos\frac{n}{2}ky \,+\, \sin\frac{n}{2}ky\,) \\
&ii)\text{ if }\; n = 4l+3 \;\Rightarrow\; \; (\,\cos\frac{n}{2}ky \,+\, \sin\frac{n}{2}ky\,) \,\rightarrow\, -(\,\cos\frac{n}{2}ky \,-\, \sin\frac{n}{2}ky\,) \\
&(\,\cos\frac{n}{2}ky \,-\, \sin\frac{n}{2}ky\,) \,\rightarrow\, (\,\cos\frac{n}{2}ky \,+\, \sin\frac{n}{2}ky\,)
\end{aligned}
\tag{12}
$$

An important observation at this point is that there is no nontrivial even or odd parity φ under (10) or (11) if it obeys anti-periodic boundary conditions because in this case (10) or (11) effectively corresponds to a transformation of the form of $u \to \frac{\pi}{2} + u$ where $u = \frac{ky}{2}$, that is, the transformation induces a group of order four rather than a group of order two (that would be the case for $v \to \pi + v$, $v = ky$). In other words $\varphi^E = \varphi + \varphi^P + \varphi^{PP} + \varphi^{PPP}$ is even while $\varphi^O = \varphi + \varphi^{PP} - \varphi^P - \varphi^{PPP}$ is odd , where $\varphi^P = K : \varphi$, $\varphi^{PP} = K : \varphi^P$, $\varphi^{PPP} = K : \varphi^{PP}$, K stands for either of the transformations (10) or (11). However, after using the explicit form of φ in (5), one notices that both even and odd eigenvectors of K, φ^E, φ^O, are identically zero. So one can not construct the Lagrangian $\mathcal{L}_k$, in particular the terms that are quadratic in φ e.g. kinetic terms, out of linear combinations of φ^E and φ^O. However one observes from (12) that the quadratic terms that are even or odd under K may be written in the form, $\varphi\varphi \pm \varphi^P\varphi^P$ or $\varphi\varphi^P \pm \varphi^P\varphi$. In fact this is the reason for adopting anti-periodic boundary conditions for y_1 and y_2. As we will see S_{fk1} in (20) should contain only off-diagonally coupled Kaluza-Klein modes and S_{fk1} should vanish after integration over extra dimension to use these modes for the Pauli-Villars like regularization at the length scales smaller than the size of the extra dimensions. This, in turn, requires the n_1 and m_1 should be summed as in the $\cos\frac{n_1+m_1}{2}k_1y_1$ term in (20). In other words it requires φ and φ^P be in the either of the combinations $\varphi\varphi + \varphi^P\varphi^P$ or $\varphi\varphi^P - \varphi^P\varphi$ i.e. in a combination of the form $i\bar{\varphi}_{n_1}\gamma^\mu\partial_\mu\varphi_{m_1}(\cos\frac{n_1}{2}k_1y_1\cos\frac{m_1}{2}k_1y_1 - \sin\frac{n_1}{2}k_1y_1\sin\frac{m_1}{2}k_1y_1)$ as done in (17-20). Such a combination can be enforced only if the φ transforms as in (12) i.e. if the corresponding dimensions obey anti-periodic boundary conditions. A similar argument is true for the boundary conditions for y_2. The boundary conditions in the direction of y_2 should be anti-periodic as well in order to make S_{fk2} in (28) to be non-vanishing after integration over extra dimensions. On the other hand the role of y_3 is only to enable the modification of the volume element on the brane $k_1y_1 = k_3y_3$ to induce the usual fermions through S_{fk2}. Taking anti-periodic boundary conditions for y_3 is unnecessary and only causes complications such as a possible mass term (of order of the inverse size of the dimension y_3) for the lowest mode in the direction of y_3 (that to be identified by the usual fermions) on contrary to the

phenomenology. These observations will be used to construct the actions S_{fk1} and S_{fk2} in the following paragraphs. Next I write down φ^P explicitly for later reference,

$$\varphi^P(x,z) = \sum_{|n|=1}^{\infty} \{ \pm f_{|n|}[\cos(\frac{|n|kz}{2}) - \sin(\frac{|n|kz)}{2})] \mp g_{|n|}[\cos(\frac{|n|kz)}{2}) + \sin(\frac{|n|kz)}{2})] \}\varphi_{|n|}(x) \tag{13}$$

where + and − in ± stands for $n = 4p+1$ and $n = 4p+3$, respectively while − and + in ∓ stands for $n = 4p+1$ and $n = 4p+3$, respectively. I also note the following relation for later reference,

$$\partial_z\varphi(x,z) = \begin{array}{lll} \frac{k}{2}\sum_{|n|=1}^{\infty}|n|\,\varphi_n^P & \text{if} & n=4p+1 \\ -\frac{k}{2}\sum_{|n|=1}^{\infty}|n|\,\varphi_n^P & \text{if} & n=4p+3 \end{array} \tag{14}$$

$$\partial_z\varphi^P(x,z) = \begin{array}{lll} -\frac{k}{2}\sum_{|n|=1}^{\infty}|n|\,\varphi_n & \text{if} & n=4p+1 \\ +\frac{k}{2}\sum_{|n|=1}^{\infty}|n|\,\varphi_n & \text{if} & n=4p+3 \end{array} \tag{15}$$

$$\varphi = \sum_{|n|=1}^{\infty}\varphi_n \quad , \quad \varphi^P = \sum_{|n|=1}^{\infty}\varphi_n^P$$

where the explicit forms of φ_n and φ_n^P are evident from (5) and (13).

III. THE MODEL

Once the background of the model is studied we are ready to formulate the model now. The space employed in this scheme is the one given in (1). I particularize the analysis to fermionic fields, and replace φ by χ. In this paper the action in the bulk will be taken to be invariant under the separate applications of the transformations (3) (and (6)) and (10) while it is broken on the brane $y_1 = y_3$ by a small amount. On the other hand the action on the brane will be taken to be invariant under the separate (and the simultaneous) applications of (10) and (11), and the simultaneous combined application (3), (4) (and (6), (7)). These symmetries together with anti-periodic boundary conditions in the 5th, 6th directions and periodic boundary conditions in the 7th direction will lead to a model with an inbuilt Pauli-Villars regularization scheme as we will see in the following paragraphs.

A. The Spectrum at the Scales Larger than the Sizes of Extra Dimensions

First we consider the 4-dimensional part of the kinetic term (except the spin connection term) of (2) for fermion fields. I require the action be invariant under (3) and (10). I consider

the zero mode in the y_3 direction in the following and take the other modes be very heavy. In other words I assume only the zero mode of y_3 be relevant to the phenomenology at the present relatively low energies that can be produced in current or near future accelerators. Then the requirement of the action be invariant under (3) (and (9)) implies the Kaluza-Klein (KK) modes with $n = 4p + 1$ couples to KK modes with $n = 4l + 3$, $p, l = 0, 1, 2, ..$ in the Lagrangian terms that are quadratic in χ and χ^P (e.g. in the kinetic terms). Then in the light of the discussion after (12) and the symmetry (3) (and (6)) the requirement of invariance of the quadratic terms under (10) requires the corresponding action be

$$S_{fk1} = \int d^4x\, d^3y\, \cos^3 k_2 y_2\, \cos k_3 y_3 \frac{1}{2}[\mathcal{L}_{fk11} + \mathcal{L}_{fk12}] + H.C. \tag{16}$$

$$\mathcal{L}_{fk11} = \frac{i}{4}[(\bar{\chi}_{(1)}\gamma^\mu\,\partial_\mu\chi_{(3)} + \bar{\chi_{(1)}}^P\gamma^\mu\,\partial_\mu\chi^P_{(3)}) + y_1 \to -y_1] \tag{17}$$

$$\mathcal{L}_{fk12} = \frac{i}{4}[(\bar{\chi}\gamma^\mu\,\partial_\mu\chi^P - \bar{\chi}^P\gamma^\mu\,\partial_\mu\chi) + (y_1 \to -y_1)] \tag{18}$$

After inserting the explicit forms of χ and χ^P (by using (5) and (13)) one finds

$$\begin{aligned} \mathcal{L}_{fk1} &= \frac{1}{2}[\mathcal{L}_{fk11} + \mathcal{L}_{fk12}] \\ &= \sum_{n_1,m_1=1}^{\infty} A^{(1,3)}_{n1,m1}\, i\bar{\chi}_{n_1}(x,y)\gamma^\mu\partial_\mu\chi_{m_1}(x,y)\, \cos\frac{n_1+m_1}{2}k_1y_1 + H.C. \end{aligned} \tag{19}$$

$$\begin{aligned} S_{fk1} &= \int d^4x\, d^2y\, \cos^3 k_2y_2\, \cos k_3y_3 \sum_{n_1,m_1=1}^{\infty} A^{(1,3)}_{n1,m1}\, i\bar{\chi}_{n_1}(x,y)\gamma^\mu\partial_\mu\chi_{m_1}(x,y) \\ &\times \int dy_1\, \cos\frac{n_1+m_1}{2}k_1y_1 + H.C. = 0 \end{aligned} \tag{20}$$

$$A^{(1,3)}_{n1,m1} = (f^*_{n1}g_{m1} + g^*_{n1}f_{m1} + f^*_{n1}f_{m1} - g^*_{n1}g_{m1})$$

where $y = y_2, y_3$ in general, and $y = y_2$ for the zero mode in the direction of y_3. Here the upper index $*$ denotes complex conjugate, $H.C.$ stands for Hermitian conjugate, and f_n, g_n's are those given in (5). The subscripts (1), (3) above refer to the modes with $n = 4p+1$ and $n = 4p+3$, respectively; and the superscript $(1,3)$ denotes that one of the subscripts n_1, m_1 is given by $4p_1 + 1$ while the other by $4s_1 + 3$, where $p, s = 0, 1, 2, ...$ The $y_1 \to -y_1$ terms in the above equations stand for the terms obtained from the preceding ones by replacing y_1's in that term by $-y_1$ and insures the invariance of the Lagrangian $\mathcal{L}_{fk1}$ under (3). The values of n_1, m_1 in (18,20) are fixed by the requirement of invariance under (10), (6), and are given by

$$n_1 = 4l_1 + 1\ ,\quad m_1 = 4p_1 + 3 \quad \text{or vice versa} \qquad l_1, p_1 = 0, 1, 2, ... \tag{21}$$

It is evident that the integration in (20) results in zero because $\int_0^{L_1} \cos\frac{n_1+m_1}{2} k_1 y_1\, dy_1 = 0$ since $n_1 + m_1 \neq 0$.

On the hyper-surface $k_3 y_3 = k_1 y_1$, I assume the symmetry (3) (and (6)) is broken by a small amount while there is an unbroken symmetry under the separate (and simultaneous) applications of (10), (11), and under the simultaneous application of (3) and (4) (and (6) and (7)). Then in addition to (16) there are additional terms given by

$$
\begin{aligned}
S_{fk2} &= \epsilon \int d^4x\; d^3y\, \delta(k_3 y_3 - k_1 y_1) \cos^3 k_2 y_2\, \cos k_3 y_3 \frac{1}{2}[\mathcal{L}_{fk21} + \mathcal{L}_{fk22}] + H.C. \qquad (22) \\
\mathcal{L}_{fk21} &= \frac{i}{8}[(\bar{\chi}_{(1,3)}\gamma^\mu\, \partial_\mu \chi_{(1,3)} + \bar{\chi}^{P1,P2}_{(1,3)}\gamma^\mu\, \partial_\mu \chi^{P1,P2}_{(1,3)} - \bar{\chi}^{P1}_{(1,3)}\gamma^\mu\, \partial_\mu \chi^{P1}_{(1,3)} - \bar{\chi}^{P2}_{(1,3)}\gamma^\mu\, \partial_\mu \chi^{P2}_{(1,3)}) \\
&+ (y_{1,2} \to -y_{1,2})] \qquad (23) \\
\mathcal{L}_{fk22} &= \frac{i}{8}[(\bar{\chi}_{(1,3)}\gamma^\mu\, \partial_\mu \chi^{P1}_{(1,3)} + \bar{\chi}^{P1}_{(1,3)}\gamma^\mu\, \partial_\mu \chi_{(1,3)} - \bar{\chi}^{P2}_{(1,3)}\gamma^\mu\, \partial_\mu \chi^{P1,P2}_{(1,3)} - \bar{\chi}^{P1,P2}_{(1,3)}\gamma^\mu\, \partial_\mu \chi^{P2}_{(1,3)} \\
&+ \bar{\chi}_{(1,3)}\gamma^\mu\, \partial_\mu \chi^{P2}_{(1,3)} + \bar{\chi}^{P2}_{(1,3)}\gamma^\mu\, \partial_\mu \chi_{(1,3)} - \bar{\chi}^{P1}_{(1,3)}\gamma^\mu\, \partial_\mu \chi^{P1,P2}_{(1,3)} - \bar{\chi}^{P1,P2}_{(1,3)}\gamma^\mu\, \partial_\mu \chi^{P1}_{(1,3)} + \bar{\chi}^{P1}_{(1,3)}\gamma^\mu\, \partial_\mu \chi^{P2}_{(1,3)} \\
&+ \bar{\chi}^{P2}_{(1,3)}\gamma^\mu\, \partial_\mu \chi^{P1}_{(1,3)} + \bar{\chi}_{(1,3)}\gamma^\mu\, \partial_\mu \chi^{P1,P2}_{(1,3)} + \bar{\chi}^{P1,P2}_{(1,3)}\gamma^\mu\, \partial_\mu \chi_{(1,3)}) + (y_{1,2} \to -y_{1,2})] \qquad (24)
\end{aligned}
$$

where $\epsilon << 1$ is some constant that accounts for the breaking of the symmetry (3) by a small amount. The superscripts $P1$, $P2$ refer to the χ's transformed under (10), (11), respectively. The subscripts $(1,3)$ refer to $n_1, m_1 = 4p_1 + 1$ and $n_2, m_2 = 4p_2 + 3$, $p_1, p_2 = 0, 1, 2, ...$ After replacing the fields χ, χ^P one finds

$$
\begin{aligned}
S_{fk2} &= \frac{\epsilon L_3}{4\pi} \sum_{n_1,m_1=1}^{\infty} \sum_{n_2,m_2=1}^{\infty} A^{(1,1)}_{n1,m1} A^{(3,3)}_{n2,m2} \int d^4x\, i\bar{\chi}_{n_1,n_2}\gamma^\mu \partial_\mu \chi_{m_1,m_2} \\
&\times \int dy_1 [\cos(\frac{n_1+m_1}{2} - 1)k_1 y_1 + \cos(\frac{n_1+m_1}{2} + 1)k_1 y_1] \\
&\times \int dy_2 [3\cos(\frac{n_2+m_2}{2} - 1)k_2 y_2 + 3\cos(\frac{n_2+m_2}{2} + 1)k_2 y_2 \\
&+ \cos(\frac{n_2+m_2}{2} - 3)k_2 y_2 + \cos(\frac{n_2+m_2}{2} + 3)k_2 y_2] \qquad (25) \\
A^{(1,1)}_{n1,m1} &= (f^*_{n1} g_{m1} + g^*_{n1} f_{m1} + f^*_{n1} f_{m1} - g^*_{n1} g_{m1}) \\
\tilde{A}^{(3,3)}_{n2,m2} &= (f^*_{n2} g_{m2} + g^*_{n2} f_{m2} + f^*_{n2} f_{m2} - g^*_{n2} g_{m2})
\end{aligned}
$$

The superscript $(1,1)$ in (25) refers to the fact that the modes with $n_1 = 4p_1 + 1$ couple to those with $m_1 = 4l_1 + 1$ while the superscript $(3,3)$ refers to that the modes with $n_2 = 4p_2 + 3$ couple to the modes with $m_2 = 4l_2 + 3$. In other words the values of n_1, m_1, n_2, m_2 are fixed by the requirement of invariance under (10), (11), (9), and are given by

$$
n_1 = 4p_1 + 1\,, \quad m_1 = 4l_1 + 1\,, \quad n_2 = 4p_2 + 3\,, \quad m_2 = 4l_2 + 3
$$

$$\text{or} \quad n_1 = 4p_1 + 3 \ , \quad m_1 = 4l_1 + 3 \quad , \quad n_2 = 4p_2 + 1 \ , \quad m_2 = 4l_2 + 1 \qquad (26)$$

$$l_1, p_1 = 0, 1, 2, ...$$

Due to the periodicity of the cosine functions (25) is non-zero after integration over extra dimensions only when the argument of the cosines in (25) are zero, that is, when

$$n_1 + m_1 - 2 = (4l_1 + 1) + (4p_1 + 1) - 2 = 0 \quad \Rightarrow \quad l_1 = p_1 = 0 \ \Rightarrow \quad n_1 = m_2 = 1$$
$$n_2 + m_2 - 6 = (4l_2 + 3) + (4p_2 + 3) - 6 = 0 \ \Rightarrow \ l_2 = p_2 = 0 \ \Rightarrow \quad n_2 = m_2 = 3 \quad (27)$$

So the integral in (25) gives

$$S_{fk2} = \frac{\epsilon L_1 L_2 L_3}{4\pi}(f_1^* g_1 + g_1^* f_1 + f_1^* f_1 - g_1^* g_1)(f_3'^* g_3' + g_3'^* f_3' + f_3'^* f_3' - g_3'^* g_3') \int d^4x \, i\bar{\chi}_{13}\gamma^\mu \partial_\mu \chi_{13} \qquad (28)$$

where the primes on f_3', g_3' are introduced to point out that these are the Fourier expansion coefficients in y_2 direction while f_1, g_1 here are the Fourier expansion coefficients in y_1 direction.

In other words at energies smaller than $\sim \frac{1}{L_{1(2)}}$ only χ_{13} is observed. Further if $n_3 = 0$ is identified by the usual particles (and the other modes in the y_3 direction are assumed to be very heavy) then χ_{130} (where $n_3 = 0$ is the zero mode corresponding to y_3 direction) is the only particle observed at present energies and it is identified by a usual (standard model) fermion. Note that the higher modes $\chi_{n_1,n_2,0}$ will not be observed at length scales larger then $L_{1(2)}$ even when they are somehow produced on contrary to the usual way of getting rid of higher Kaluza-Klein modes by taking them very massive (compared to the energy scales attainable at current experiments). Moreover the matter action (hence the Lagrangian) is multiplied by the small parameter ϵ at scales larger than the size of the extra dimensions. This may explain why gravitational force is so smaller than the other forces since the Lagrangian enters the Einstein equations through energy-momentum tensor. Another point worth to mention is that the dimension of the space employed here (i.e 7) is the minimum dimension that this scheme can be applied as is evident from the argument given in the preceding paragraphs. Fifth dimension, y_1 is necessary to make the contribution due to S_{fk1} (that is used for regularization at smaller length scales) be vanishing at relatively large length scales by the requirement of the invariance of the action under (3) and (10). The sixth dimension, y_2 is necessary to induce the non-zero $i\bar{\chi}_{13}\gamma^\mu \partial_\mu \chi_{13}$ in (28) by the requirement of the invariance of the action under the separate (and simultaneous) applications of (10), (11),

and the simultaneous application of (3) and (4) (and (6 and (7)). The role of the seventh dimension, y_3 is to change the factor $\cos k_3 y_3$ in the volume element to $\cos k_1 y_1$ by the delta function so that the non-zero contribution to S_{fk2} at large length scales through the diagonal term $i\bar{\chi}_{13}\gamma^\mu \partial_\mu \chi_{13}$ may be induced. In fact this also explains why the transformations (3), (4), (10), (11) do not act on y_3. The only role of y_3 is to change the form of the volume element so that the diagonal term $i\bar{\chi}_{13}\gamma^\mu \partial_\mu \chi_{13}$ in (28) is induced. A non-trivial transformation of y_3 under these transformation would only make the model more complicated.

Next consider a possible mass for $\chi_{130}(x)$ that may be induced by the extra dimensional pieces of the kinetic terms in S_{fk1} and S_{fk2}. First consider the spin connection terms of the form $\bar{\chi}\Gamma^A \omega_A \chi$ where $\omega_A = (e^a_B \partial_A e^{Bb} + e^a_B e^{Cb}\Gamma^B_{CA})[\gamma^a, \gamma^b]$, $(A, B, C, a, b = 0, 1, 2, 3, 5, 6, 7)$. Although the spin connection terms do not have the form of a mass term in the simplest scheme where f_n, g_n are simple real numbers one may obtain mass terms if we allow a more general form for f_n, g_n, which is compatible with the 4-dimensional local Lorentz invariance, that is,

$$f_n = f_{0n} + \sum \Gamma^a f_{a\,n} + \sum \Gamma^a \Gamma^b f_{ab\,n} + \sum \Gamma^a \Gamma^b \Gamma^c f_{abc\,n}\ , \quad \{\Gamma^a, \Gamma^b\} = 2g^{ab}\ , \quad a, b, c = 5, 6, 7 \quad (29)$$

where a similar expression may be written for g_n as well. The non-zero values of the spin connection ω_A are $\omega_5 \propto \frac{\tan k_2 y_2}{\cos k_3 y_3}$, $\omega_6 \propto \sin k_3 y_3$, $\omega_7 \propto \cos^2 k_3 y_3 \sin k_3 y_3$. After integrating over y_1, y_2, y_3 they give zero for both of the terms of the form S_{fk1} and S_{fk2} because the symmetries (6, 7) set the overall extra dimensional contribution from $\bar{\chi}\Gamma^A \chi$ to be a cosine while the spin connection terms contain one sine term so that the overall extra dimensional contribution is a sine that gives zero after integration over extra dimensions. So spin connection terms are already are not relevant for the 4-dimensional mass terms. Next consider a possible mass term that may be induced by the extra dimensional derivative terms of the form $\bar{\chi}\Gamma^a \partial_a \chi$ where $a = 5, 6, 7$ ($x_5 = y_1$, $x_6 = y_2$, $x_7 = y_3$) in the kinetic terms. It is evident from (14) and (15) that the extra dimensional pieces of the kinetic terms (due to derivatives) in the Lagrangian does not obey the symmetry under the simultaneous application of (3) and (4). So they are not allowed. In fact explicit evaluation of these terms give zero identically due to the same reason as the null contribution of the spin connection term to mass. So no masses are induced due to the extra dimensional piece of the kinetic terms. However one may introduce a mass through a term $m\bar{\chi}\chi$ (or a fermion-Higgs interaction term $m\bar{\chi}\phi\chi$) on the brane $k_1 y_1 = k_3 y_3$ in the same as done for getting the

$\bar{\chi}_{13}\gamma^\mu\partial_\mu\chi_{13}$ in (28).

B. General Considerations on the Spectrum

Before discussing the field spectrum at the scales smaller than the sizes of extra dimensions I write the fields and the Lagrangian in a simpler form so that the discussion of the following parts becomes simpler. The results obtained in this subsection will especially be important for the discussion of the field spectrum at the scales smaller than the sizes of extra dimensions while the results hold for all scales. However the results obtained here are not crucial for the discussion of the preceding subsection. Moreover the results of the preceding subsection will be used to clarify the general statements given here. So this is the right point to discuss the results obtained here.

In the general simple Kaluza-Klein (KK) prescription all Kaluza-Klein modes correspond to distinct elementary particles that are independent of each other (except sharing the same internal properties). On the other hand the KK modes in this scheme in general mix with each other through off-diagonal couplings in the kinetic terms. It is evident from the discussion given in the preceding subsection that the modes in this scheme belong to four different sets. The sets (for $n_3 = 0$) are; i) $n_1 = 4p_1 + 1$, $n_2 = 4p_2 + 1$, ii) $n_1 = 4p_1 + 1$, $n_2 = 4p_2 + 3$, iii) $n_1 = 4p_1 + 3$, $n_2 = 4p_2 + 1$, i) $n_1 = 4p_1 + 3$, $n_2 = 4p_2 + 3$. The sets that are relevant for us are ii), iii), iii) because we are concerned with the modes that couple to ψ_{130} (i.e. to the usual standard model fermion in this construction). The modes in each set can not be identified with distinct physical fields because all modes in the set are entangled. So each set must be identified with a particular field (or particle). At this point I make two plausible assumptions to simplify the analysis. I take the extra dimensions be related to the internal properties of fields. I also take the internal properties of the fields be independent of their 4-dimensional coordinates. These two conditions greatly simplify the Fourier decomposition of a field corresponding to one of these sets. To be specific, for example, consider the modes with $n_1 = 4p_1 + 1$. The corresponding Fourier decomposition may be written as

$$\chi_{(1)}\,(x,y) \;=\; \sum_{p=0}^{\infty}[\,a^{(1)}_{4p+1}(y_2,y_3)\cos\frac{4p+1}{2}k_1y_1 \;+\; b^{(1)}_{4p+1}(y_2,y_3)\sin\frac{4p+1}{2}k_1y_1\,]\chi^{(1)}_n(x^\mu) \quad (30)$$

In order to be able to satisfy the condition that a field at different 4-dimensional coordinates

has the same internal properties the composition in (30) should reduce to the following form

$$\chi_{(1)}(x,y) = \chi_1(x)\,F(y) \tag{31}$$
$$F(y) = \sum_{p=0}^{\infty}[\,a_{4p+1}^{(1)}(y_2,y_3)\cos\frac{4p+1}{2}k_1y_1 \,+\, b_{4p+1}^{(1)}(y_2,y_3)\sin\frac{4p+1}{2}k_1y_1\,] \tag{32}$$

provided that extra dimensions are identified with internal properties of particles. The consideration of this argument in a more concrete form through the study of S_{fk1}, S_{fk2} may be more instructive. So I give such an analysis below.

First consider S_{fk1}. The general form of S_{fk1} is

$$S_{fk1} \sim \int \cos^3 k_2y_2\,\cos k_3y_3\,[i\bar{\chi}_{(1)}\gamma^\mu\,\partial_\mu\chi_{(3)} + 1 \leftrightarrow 3]\;d^4x\;d^3y \tag{33}$$

where the terms with the upper indices P in (17,18) are skipped because the complete form of S_{fk1} is not necessary for the analysis given here. The simple form given in (33) is enough to see the essential points in this subsection. Here

$$\chi_{(1)}(x,y) = \sum_{p=0}^{\infty}[\,a_{4p+1}^{(1)}(y_2,y_3)\cos\frac{4p+1}{2}k_1y_1 \,+\, b_{4p+1}^{(1)}(y_2,y_3)\sin\frac{4p+1}{2}k_1y_1\,]\chi_n^{(1)}(x^\mu) \tag{34}$$
$$\chi_{(3)}(x,y) = \sum_{p=0}^{\infty}[\,a_{4p+3}^{(3)}(y_2,y_3)\cos\frac{4p+3}{2}k_1y_1 \,+\, b_{4p+3}^{(3)}(y_2,y_3)\sin\frac{4p+3}{2}k_1y_1\,]\chi_n^{(3)}(x^\mu) \tag{35}$$

After considering (33) and (34,35) one observes that all modes in $\chi_{(1)}$ are mixed with each other and the same is true for $\chi_{(3)}$. So it is impossible to entangle the modes in these states as different particles. In other words one should treat $\chi_{(1)}$ or $\chi_{(3)}$ as a single entity. Extra dimensions are related to the internal properties of the elementary particles. Elementary particles at different 4-dimensional coordinates do not have different internal properties (at least at the scales reached by current experiments). If we assume this to hold at all scales in the 4-dimensional coordinates it implies that

$$\chi_n^{(1)} = \chi_m^{(1)} \quad , \quad \chi_n^{(3)} = \chi_m^{(3)} \qquad \text{for all } n,m \tag{36}$$

Then (34) and (35) become

$$\chi_{(1)}(x,y) = \chi_1(x)\,F(y) \tag{37}$$
$$F(y) = \sum_{p=0}^{\infty}[\,a_{4p+1}^{(1)}(y_2,y_3)\cos\frac{4p+1}{2}k_1y_1 \,+\, b_{4p+1}^{(1)}(y_2,y_3)\sin\frac{4p+1}{2}k_1y_1\,] \tag{38}$$
$$\chi_{(3)}(x,y) = \chi_3(x)\,G(y) \tag{39}$$
$$G(y) = \sum_{p=0}^{\infty}[\,a_{4p+3}^{(3)}(y_2,y_3)\cos\frac{4p+3}{2}k_1y_1 \,+\, b_{4p+3}^{(3)}(y_2,y_3)\sin\frac{4p+3}{2}k_1y_1\,] \tag{40}$$

In a similar way the general form of S_{fk2} is

$$S_{fk2} \sim \frac{\epsilon}{k_3} \int \cos^3 k_2 y_2 \, \cos k_1 y_1 \delta(x_3 - x_1) \, [i\bar{\tilde{\chi}}_{(1,3)} \gamma^\mu \, \partial_\mu \tilde{\chi}_{(1,3)} + i\bar{\tilde{\chi}}_{(3,1)} \gamma^\mu \, \partial_\mu \tilde{\chi}_{(3,1)}] \, d^4x \, d^3y \quad (41)$$

Here

$$\begin{aligned}
\tilde{\chi}_{(r,q)}\,(x,y) \;&=\; \chi_{(r,q)}(x)\, \tilde{F}_r(y_1,y_3) \tilde{G}_q(y_2,y_3) && (42)\\
\tilde{F}_r(y_1,y_3) \;&=\; \sum_{p=0}^{\infty} [\, \tilde{a}^{(1)}_{4p+r}(y_3) \cos \frac{4p+r}{2} k_1 y_1 \;+\; \tilde{b}^{(1)}_{4p+r}(y_3) \sin \frac{4p+r}{2} k_1 y_1 \,] && (43)\\
\tilde{G}_q(y_2,y_3) \;&=\; \sum_{p=0}^{\infty} [\, \tilde{a}^{(3)}_{4p+q}(y_3) \cos \frac{4p+q}{2} k_2 y_2 \;+\; \tilde{b}^{(3)}_{4p+q}(y_3) \sin \frac{4p+q}{2} k_2 y_2 \,] && (44)\\
& \quad r,q = 1,3 \;, \quad r \neq q
\end{aligned}$$

where a_n's, b_n's in (38,40) are changed into $\tilde{a}_n$'s, $\tilde{b}_n$'s since the fields in this case are confined into a subspace of the whole space, and all modes in the direction of y_2 contribute to $\chi_{(1)}$ in S_{fk1} while only the modes with $n_2 = 4p_2 + 1$ contribute to S_{fk2} when $n_1 = 4p_1 + 3$ and the modes $n_2 = 4p_2 + 3$ contribute when $n_1 = 4p_1 + 1$.

C. The spectrum at the scales smaller than the sizes of extra dimensions

After the study of the technical points given above we return to the main discussion. Now see what happens when one goes to the scales smaller than the size of the extra dimensions $L_{1(2)}$. In scales smaller than the size of $L_{1(2)}$ all Kaluza-Klein modes in the corresponding direction $y_{1(2)}$ are observed since conformal factors $\cos k_1 y_1$ and $\cos k_2 y_2$ can not hide these modes any more. However it has been shown in the preceding section that these modes for $n_3 = 0$ reduce to $\chi_{130}(x)$, $\chi_{110}(x)$, $\chi_{310}(x)$, $\chi_{330}(x)$ from a 4-dimensional point of view under the assumptions that extra dimensions can be identified by internal properties of fields and internal properties of particles are the same at all 4-dimensional coordinates (once their extra dimensional coordinates are kept fixed). In the light of above analysis let us consider the kinetic term of χ_{130} in the scales smaller than the sizes of the extra dimensions.

1. Outside the brane $k_1y_1 = k_3y_3$

At the points $k_1y_1 \neq k_3y_3$ the only contribution is due to (19) where in the light of the preceding section the Lagrangian may be written as

$$\begin{aligned}
\mathcal{L}_{fk1} &= \sum_r i\bar{\chi}_{130}(x)\gamma^\mu\,\partial_\mu\chi_{3r0}(x)\sum_{p,s=0}^{\infty}\tilde{A}^{(r)}_{ps}(y_2)\cos 2(p+s+1)k_1y'_1 \qquad (45)\\
\tilde{A}^{(r)}_{ps}(y_2) &= f^{(3)*}_{4p+3}(y_2)g^{(r)}_{4s+r}(y_2)+g^{(3)*}_{4p+3}(y_2)f^{(r)}_{4s+r}(y_2)+f^{(3)*}_{4p+3}(y_2)f^{(r)}_{4s+r}(y_2)-g^{(3)*}_{4p+3}(y_2)g^{(r)}_{4s+r}(y_2)
\end{aligned}$$

where, on contrary to (17-20), the y_2 dependence is expressed explicitly and $r = 1,3$ stands for the modes $n_2 = 4p_2 + 1$, $n_2 = 4p_2 + 3$, respectively; and the upper indices $(1,3)$ in (19) are suppressed here because the fact that the modes $n_1 = 4p_1 + 1$ couple to the modes $m_1 = 4s_1 + 3$ is evident in (45). Eq.(45) may be written in the form

$$\mathcal{L}_{fk1} = \frac{i}{2}\lim_{x'\to x}\partial_\mu\ \left(\bar{\chi}_{130}(x'),\bar{\chi}_{310}(x')\bar{\chi}_{330}(x')\right)\mathbf{M}\,\gamma^\mu\begin{pmatrix}\chi_{130}(x)\\ \chi_{310}(x)\\ \chi_{330}(x)\end{pmatrix} \qquad (46)$$

Here

$$\mathbf{M} = \begin{pmatrix}0 & \mathcal{B} & \mathcal{C}\\ \mathcal{B} & 0 & 0\\ \mathcal{C} & 0 & 0\end{pmatrix} \qquad (47)$$

$$\mathcal{B} = \sum_{p,s=0}^{\infty}\tilde{A}^{(1)}_{ps}\cos 2(p+s+1)k_1y'_1 \quad , \quad \mathcal{C} = \sum_{p,s=0}^{\infty}\tilde{A}^{(3)}_{ps}\cos 2(p+s+1)k_1y'_1$$

The diagonalization of $\mathbf{M}$ in (47) results in

$$\mathcal{L}_{fk1} = iB(y)[\bar{\psi}_1(x)\gamma^\mu\,\partial_\mu\psi_1(x) - \bar{\psi}_2(x)\gamma^\mu\,\partial_\mu\psi_2(x)] \qquad (48)$$

$$\psi_1 = \frac{1}{\sqrt{2}}[\chi_{130} + (\cos\theta\chi_{310} + \sin\theta\chi_{330})] \qquad (49)$$

$$\psi_2 = \frac{1}{\sqrt{2}}[\chi_{130} - (\cos\theta\chi_{310} + \sin\theta\chi_{330})] \qquad (50)$$

$$\cot\theta = \frac{\mathcal{B}}{\mathcal{C}} \quad , \quad B(y) = \frac{1}{4}\sqrt{(\mathcal{B}^2 + \mathcal{C}^2)} \quad , \quad y = y_1, y_2 \qquad (51)$$

There is another state $\psi_3 = \sin\theta\chi_{310} - \cos\theta\chi_{330}$ but this does not contribute to (48). So it is an auxiliary field. Although the sign of the kinetic term of ψ_2 in (48) is opposite of a usual fermion (and so it is a ghost-like field) it does not suffer from the problems of the usual

ghosts. ψ_1 or ψ_2 in (48) can not be introduced or removed from (48) separately because (48) follows from the couplings of χ_{130}, χ_{310}, χ_{330}. So ψ_1, ψ_2 form a single system. For example in this case ψ_1, ψ_2 may be considered as the components of a single field with a 8-component spinor and the gamma matrices given by $\gamma^\mu \odot \tau_3$ where $\odot$ denotes tensor product and τ_3 is the third Pauli matrix. This solves the problem of negative norm for ψ_2 because there is single norm i.e. that of the system composed of ψ_1, ψ_2. Moreover since ψ_1 and ψ_2 have the same internal space properties and they form a single system they may be assigned the same 4-momentum with positive energy, and this solves the negative energy problem of ψ_2. However the extension of this argument to the fields other than the fermions is not straightforward and requires additional study.

2. On the brane $k_1 y_1 = k_3 y_3$

On the brane $k_1 y_1 = k_3 y_3$ one should include the effect of $\mathcal{L}_{fk2}$ as well. To see the situation near $k_1 y_1 = k_3 y_3$ I consider a tiny patch on the brane and integrate $\mathcal{L}_{fk1}$, $\mathcal{L}_{fk2}$ over that patch while the dependence on x^μ and y_2 are point-wise. Then I form a mixing matrix similar to (47) to the study the resulting spectrum. For convenience I change the parameters y_1, y_3 to $u = k_1 y_1 - k_3 y_3$, $v = k_1 y_1 + k_3 y_3$. I take the patch to be a rectangular area on the line $k_1 y_1 = k_3 y_3$ and on its neighborhood, of the width and length 2Δ and Δ', given by

$$-\Delta \leq u \leq \Delta\ , \quad v \leq v' \leq v + \Delta' \quad , \quad u = k_1 y_1 - k_3 y_3\ , \quad v = k_1 y_1 + k_3 y_3 \tag{52}$$

From a 4-dimensional perspective the effective Lagrangian at the scales smaller than the size of the extra dimensions may be taken as the original Lagrangian times the extra dimensional conformal factors in the volume element. The inclusion of these terms in the volume element is essential because the $\cos k_3 y_3$ term in the volume element is effected by the integration over y_3 due to the δ function in $\mathcal{L}_{fk2}$. So I consider the extra dimensional terms in the volume element times the Lagrangian in the following. The corresponding term for $\mathcal{L}_{fk1}$ is

$$\begin{aligned} & i \sum_r \bar{\chi}_{130}(x) \gamma^\mu \partial_\mu \chi_{3r0}(x) \cos^3 k_2 y_2 \sum_{p,s=0}^{\infty} A_{ps}^{(r)}(y_2) \\ & \times \int_v^{v+\Delta'} dv' \int_{-\Delta}^{\Delta} du \cos \frac{1}{2}[2(p+s)+1](v'+u) \cos \frac{1}{2}(v'-u) \\ = \ & \frac{1}{2} \sum_r i \bar{\chi}_{130}(x) \gamma^\mu \partial_\mu \chi_{3r0}(x) \cos^3 k_2 y_2 A_{ps}^{(r)}(y_2) \end{aligned}$$

$$
\begin{aligned}
& \times \int_v^{v+\Delta'} dv' \int_{-\Delta}^{\Delta} du\{\cos[(p+s)u+(p+s+1)v'] + \cos[(p+s+1)u+(p+s)v']\} \\
= & \sum_r i\bar{\chi}_{130}(x)\gamma^\mu\,\partial_\mu\chi_{3r0}(x)\ \cos^3 k_2y_2 A_{ps}^{(r)}(y_2) \\
& \times \frac{1}{(p+s)(p+s+1)}[\sin(p+s)\Delta\ \sin(p+s+1)v'\ |_v^{v+\Delta'} + \sin(p+s+1)\Delta\ \sin(p+s)v'\ |_v^{v+\Delta'}] \\
\simeq & \sum_r i\bar{\chi}_{130}(x)\gamma^\mu\,\partial_\mu\chi_{3r0}(x)\ \cos^3 k_2y_2 A_{ps}^{(r)}(y_2) \\
& \times \frac{\Delta'}{(p+s)(p+s+1)}[\sin(p+s)\Delta\ \cos(p+s+1)(k_1y_1+k_3y_3) \\
& + \sin(p+s+1)\Delta\ \cos(p+s)(k_1y_1+k_3y_3)]
\end{aligned} \tag{53}
$$

Here

$$
\begin{aligned}
A_{ps}^{(r)} = & \sum_{p_2=0}^{\infty}\{\, f'_{p,|4p_2+3|}[\cos(\frac{|4p_2+3|k_2y_2}{2}) + \sin(\frac{|4p_2+3|k_2y_2)}{2})] \\
& + g'_{p,|4p_2+3|}[\cos(\frac{|4p_2+3|k_2y_2)}{2}) - \sin(\frac{|4p_2+3|k_2z_2)}{2})]\,\}^* \\
& \times \sum_{s_2=0}^{\infty}\{\, f'_{s,|4s_2+r|}[\cos(\frac{|4s_2+r|k_2y_2}{2}) + \sin(\frac{|4s_2+r|k_2y_2)}{2})] \\
& + g'_{s,|4s_2+r|}[\cos(\frac{|4s_2+r|k_2y_2)}{2}) - \sin(\frac{|4s_2+r|k_2z_2)}{2})]\,\}
\end{aligned} \tag{54}
$$

where $r=1$ or $r=3$ stands for the modes in the direction of y_2 with $n_2=4l+1$ or $n_2=4l+3$, $l=0,1,2,...$; respectively; and the primes over f_n's, g_n's do not have the same meaning as those in (28) and stand for the fact that they are in general not f_n's, g_n's themselves, rather their linear combinations. The corresponding term for $\mathcal{L}_{fk2}$ is

$$
\begin{aligned}
& i\epsilon\,\bar{\chi}_{130}(x)\gamma^\mu\,\partial_\mu\chi_{130}(x)\ \cos^3 k_2y_2 \sum_{p_1,s_1=0}^{\infty}\tilde{A}_{p_1s_1}^{(1,1)} \sum_{p_2,s_2=0}^{\infty}\tilde{A}_{p_2s_2}^{(3,3)} \\
& \times \cos[2(p_2+s_2)+3]k_2y_2 \int_v^{v+\Delta'} dv' \int_{-\Delta}^{\Delta} du \cos\frac{1}{2}[2(p_1+s_1)+1](v'+u)\,\delta(u)\ \cos\frac{1}{2}(v'-u) \\
+ & \ i\epsilon\,\bar{\chi}_{310}(x)\gamma^\mu\,\partial_\mu\chi_{310}(x)\ \cos^3 k_2y_2 \sum_{p_1,s_1=0}^{\infty}\tilde{A}_{p_1s_1}^{(3,3)} \sum_{p_2,s_2=0}^{\infty}\tilde{A}_{p_2s_2}^{(1,1)} \\
& \times \cos[2(p_2+s_2)+1]k_2y_2 \int_v^{v+\Delta'} dv' \int_{-\Delta}^{\Delta} du \cos\frac{1}{2}[2(p_1+s_1)+3](v'+u)\,\delta(u)\ \cos\frac{1}{2}(v'-u) \\
= & \ i\epsilon\,\bar{\chi}_{130}(x)\gamma^\mu\,\partial_\mu\chi_{130}(x)\ \cos^3 k_2y_2 \sum_{p_1,s_1=0}^{\infty}\tilde{A}_{p_1s_1}^{(1,1)} \sum_{p_2,s_2=0}^{\infty}\tilde{A}_{p_2s_2}^{(3,3)} \\
& \times \cos[2(p_2+s_2)+3]k_2y_2\,\frac{1}{2}\{\frac{\sin(p_1+s_1+1)v'}{p_1+s_1+1}\ |_v^{v+\Delta'} + \frac{\sin(p_1+s_1)v'}{p_1+s_1}\ |_v^{v+\Delta'}\} \\
+ & \ i\epsilon\,\bar{\chi}_{310}(x)\gamma^\mu\,\partial_\mu\chi_{310}(x)\ \cos^3 k_2y_2 \sum_{p_1,s_1=0}^{\infty}\tilde{A}_{p_1s_1}^{(3,3)} \sum_{p_2,s_2=0}^{\infty}\tilde{A}_{p_2s_2}^{(1,1)}
\end{aligned}
$$

$$
\begin{aligned}
& \times\ \cos\left[2(p_2+s_2)+1\right]k_2y_2\,\frac{1}{2}\{\frac{\sin\left(p_1+s_1+2\right)v'}{p_1+s_1+2}\ |_v^{v+\Delta'}+\frac{\sin\left(p_1+s_1+1\right)v'}{p_1+s_1+1}\ |_v^{v+\Delta'}\} \\
\simeq\ & i\epsilon\,\bar{\chi}_{130}(x)\gamma^\mu\,\partial_\mu\chi_{130}(x)\ \cos^3 k_2y_2\sum_{p_1,s_1=0}^{\infty}\tilde{A}^{(1,1)}_{p_1s_1}\sum_{p_2,s_2=0}^{\infty}\tilde{A}^{(3,3)}_{p_2s_2} \\
& \times\ \cos\left[2(p_2+s_2)+3\right]k_2y_2\,\frac{\Delta'}{2}\{\frac{\cos\left(p_1+s_1+1\right)(k_1y_1+k_3y_3)}{p_1+s_1+1}+\frac{\cos\left(p_1+s_1\right)(k_1y_1+k_3y_3)}{p_1+s_1}\} \\
+\ & i\epsilon\,\bar{\chi}_{310}(x)\gamma^\mu\,\partial_\mu\chi_{310}(x)\ \cos^3 k_2y_2\sum_{p_1,s_1=0}^{\infty}\tilde{A}^{(3,3)}_{p_1s_1}\sum_{p_2,s_2=0}^{\infty}\tilde{A}^{(1,1)}_{p_2s_2}\ \cos\left[2(p_2+s_2)+1\right]k_2y_2 \\
& \times\ \frac{\Delta'}{2}\{\frac{\cos\left(p_1+s_1+2\right)(k_1y_1+k_3y_3)}{p_1+s_1+2}+\frac{\cos\left(p_1+s_1+1\right)(k_1y_1+k_3y_3)}{p_1+s_1+1}\} \qquad (55)
\end{aligned}
$$

where

$$
\tilde{A}^{(1,1)}_{p_1s_1} = (f^{(1)*}_{4p_1+1}g^{(1)}_{4s_1+1}+g^{(1)*}_{4p_1+1}f^{(1)}_{4s_1+1}+f^{(1)*}_{4p_1+1}f^{(1)}_{4s_1+1}-g^{(1)*}_{4p_1+1}g^{(1)}_{4s_1+1}) \qquad (56)
$$

$$
\tilde{A}^{(3,3)}_{p_2s_2} = (f^{(3)*}_{4p_2+3}g^{(3)}_{4s_2+3}+g^{(3)*}_{4p_2+1}f^{(3)}_{4s_2+3}+f^{(3)*}_{4p_2+3}f^{(3)}_{4s_2+3}-g^{(3)*}_{4p_2+3}g^{(3)}_{4s_2+3}\,) \qquad (57)
$$

Here f_n and g_n's are constant. Then the total effective Lagrangian may be written as

$$
\mathcal{L}^{eff}_{fk2}\ =\ \frac{i}{2}\lim_{x'\to x}\partial_\mu\ \left(\,\bar{\chi}_{130}(x'),\bar{\chi}_{310}(x')\bar{\chi}_{330}(x')\,\right)\,\tilde{\mathbf{M}}\,\gamma^\mu\begin{pmatrix}\chi_{130}(x)\\ \chi_{310}(x)\\ \chi_{330}(x)\end{pmatrix} \qquad (58)
$$

where

$$
\tilde{\mathbf{M}}\ =\ \begin{pmatrix}\tilde{\mathcal{A}} & \tilde{\mathcal{B}} & \tilde{\mathcal{C}}\\ \tilde{\mathcal{B}} & \tilde{\mathcal{D}} & 0\\ \tilde{\mathcal{C}} & 0 & 0\end{pmatrix} \qquad (59)
$$

Here

$$
\tilde{\mathcal{A}}\ \simeq\ \epsilon\cos^3 k_2y_2\sum_{p_1,s_1=0}^{\infty}\tilde{A}^{(1,1)}_{p_1s_1}\,\tilde{T}^{(1,3)}_{p_1,s_1}(y_1)\sum_{p_2,s_2=0}^{\infty}\tilde{A}^{(3,3)}_{p_2s_2}\ \cos\left[2(p_2+s_2)+1\right]k_2y_2 \qquad (60)
$$

$$
\tilde{\mathcal{B}}\ \simeq\ \cos^3 k_2y_2\sum_{p,s=0}^{\infty}A^{(1)}_{ps}(y_2)\,T_{p,s}(y_1) \qquad (61)
$$

$$
\tilde{\mathcal{C}}\ \simeq\ \cos^3 k_2y_2\sum_{p,s=0}^{\infty}A^{(3)}_{ps}(y_2)\,T_{p,s}(y_1) \qquad (62)
$$

$$
\tilde{\mathcal{D}}\ \simeq\ \epsilon\cos^3 k_2y_2\sum_{p_1,s_1=0}^{\infty}\tilde{A}^{(3,3)}_{p_1s_1}\,\tilde{T}^{(3,1)}_{p_1,s_1}(y_1)\sum_{p_2,s_2=0}^{\infty}\tilde{A}^{(1,1)}_{p_2s_2}\ \cos\left[2(p_2+s_2)+3\right]k_2y_2 \qquad (63)
$$

where

$$
\tilde{T}^{(1,3)}_{p_1,s_1}(y_1)\ =\ \frac{\Delta'}{2}\{\frac{\cos\left(p_1+s_1+1\right)(k_1y_1+k_3y_3)}{p_1+s_1+1}+\frac{\cos\left(p_1+s_1\right)(k_1y_1+k_3y_3)}{p_1+s_1}\} \qquad (64)
$$

$$\tilde{T}^{(3,1)}_{p_1,s_1}(y_1) = \frac{\Delta'}{2}\{\frac{\cos{(p_1+s_1+2)(k_1y_1+k_3y_3)}}{p_1+s_1+2} + \frac{\cos{(p_1+s_1+1)(k_1y_1+k_3y_3)}}{p_1+s_1+1}\} \quad (65)$$

$$T_{p,s}(y_1) = \frac{\Delta'}{(p+s)(p+s+1)}[\sin{(p+s)\Delta}\,\cos{(p+s+1)(k_1y_1+k_3y_3)} + \sin{(p+s+1)\Delta}\,\cos{(p+s)(k_1y_1+k_3y_3)}] \quad (66)$$

Note that $\Delta' << 2\pi$ is employed in (60-63) since the aim is to study the small scales point-wise as much as possible (while without causing any ambiguity due to the delta function on the brane). We observe that none of the terms in $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$, $\tilde{\mathcal{D}}$ blows up as $\Delta \rightarrow 0$ or $\Delta' \rightarrow 0$ or $y_2 \rightarrow 0$ or $y_3 \rightarrow 0$. In the light of this observation denote the maximum possible values of $\frac{\tilde{\mathcal{A}}}{\epsilon}$, $\frac{\tilde{\mathcal{D}}}{\epsilon}$; and the minimum possible values of $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$ by

$$(\frac{\tilde{\mathcal{A}}}{\epsilon})_{max} = A_{mx}\ ,\quad (\frac{\tilde{\mathcal{D}}}{\epsilon})_{max} = D_{mx}\ ,\quad \tilde{\mathcal{B}}_{min} = B_{mn}\ ,\quad \tilde{\mathcal{C}}_{min} = C_{mn} \quad (67)$$

It is always possible to choose ϵ so small (i.e to choose the breaking of the symmetry (3) so small) that

$$X \gg Y \quad (68)$$

is always satisfied, where X stands for the smaller of B_{mn}, C_{mn} and Y stands for the greater of $\epsilon\,A_{mx}$, $\epsilon\,D_{mx}$. So

$$\tilde{\mathcal{A}},\tilde{\mathcal{D}} \ll \tilde{\mathcal{B}},\tilde{\mathcal{C}} \quad (69)$$

provided that ϵ is taken sufficiently small. In other words provided $\epsilon \ll 1$ is sufficiently small

$$\tilde{\mathbf{M}} \simeq \begin{pmatrix} 0 & \tilde{\mathcal{B}} & \tilde{\mathcal{C}} \\ \tilde{\mathcal{B}} & 0 & 0 \\ \tilde{\mathcal{C}} & 0 & 0 \end{pmatrix} \quad (70)$$

I take ϵ such that Eq.(70) is satisfied. Therefore the conclusions about the spectrum of the fields at the points $k_1y_1 \neq k_3y_3$ essentially remain the same at the points $k_1y_1 \simeq k_3y_3$. In fact it would be enough for us to have the relation given in (70) up to very small length scales in the order of Planck scale. In that case the length scales below this scale (where the string theory [12] or another quantum gravity scheme prevails) would be irrelevant to quantum field theory.

IV. PAULI-VILLARS-LIKE REGULARIZATION

To compare the propagators in the scales larger and smaller than the sizes of the extra dimensions one should first put $\mathcal{L}_{fk2}$ into its canonical form by dividing S_{fk2} in (28) by the factor in front of it, N given by

$$N = \frac{\epsilon L_1 L_2 L_3}{4\pi}(f_1^* g_1 + g_1^* f_1 + f_1^* f_1 - g_1^* g_1)(f_3'^* g_3' + g_3'^* f_3' + f_3'^* f_3' - g_3'^* g_3') \tag{71}$$

Note that dividing the total action by an overall constant does not change the result because it does not change the equations of motion. In general χ_{130} may have a mass m at the scales larger than the sizes of the extra dimensions (e.g. through a mass term similar to S_{fk2} in from as mentioned before). So at the scales larger than the size of the extra dimensions the general form of the propagator of χ_{130} is

$$D(p) = \frac{i}{\not p + m} \tag{72}$$

At the scales smaller than the size of extra dimensions the effective propagator follows from (48) is the sum of the propagators [13] due to ψ_1 and ψ_2; D_1, D_2

$$\begin{aligned} D_{eff}(p) &= D_1(p) + D_2(p) \sim \frac{1}{B'}\left[\frac{i}{\not p + m_1} - \frac{i}{\not p + m_2}\right] = \frac{i(m_2 - m_1)}{B'(\not p + m_1)(\not p + m_2)} \\ B' &= N\, B(y) \cos^3 k_2 y_2 \cos k_3 y_3 \end{aligned} \tag{73}$$

where m_1, m_2, in general, may depend on y_1, y_2, and I have assumed for sake of generality that ψ_1, ψ_2 may have two different effective masses, at scales smaller than the size of extra dimensions, that may be induced by spin connection terms, Higgs mechanism, or some other mechanism. Note that the internal fermion lines in a Feynman diagram must be identified by (73) rather than (72) because the off-diagonal terms in (59) are dominant for all energies (possibly at Planck scale or higher scales) as observed in (69). On the other hand the fermion external lines should be identified by χ_{130} or χ_{310} (that may come from the terms of the form, $\chi_{130} X \chi_{130}$, $\chi_{310} X \chi_{310}$, $X = \not\Omega$, ϕ where Ω_μ, ϕ are gauge, scalar fields, respectively). For the diagrams containing a single fermion internal line this scheme is quite similar to Pauli-Villars regularization. However this scheme is not wholly equivalent to Pauli-Villars regularization in the general case [8, 14]. For the Feynman diagrams containing a single fermion internal line this scheme essentially amounts to Pauli-Villars regularization when $m_1 \neq m_2$ while it amounts to finite renormalization when $m_1 = m_2$. The implications of

this scheme for higher number of fermion internal lines and its comparison with Pauli-Villars regularization needs further study. In a minimal scheme it needs, at least, the incorporation of a $\chi \not{\Omega}\chi$ or $\chi\phi\chi$ type of term into Lagrangian including the study of the effects of the higher modes of Ω_μ or ϕ. This is a quite tedious and intricate task and needs a separate study by its own. However one may see the essential lines of the regularization by (imposing periodic boundary conditions for X in all extra dimensions and) considering the zero mode of X in $\chi_{130}X\chi_{130}$ and $\chi_{310}X\chi_{310}$. Such a crude analysis suggests that this regularization is rather similar to Pauli-Villars regularization. These points need a separate study by its own and should be considered in future studies.

V. CONCLUSION

In summary I have introduced a scheme where there is a single usual particle χ_{130} at the scales larger than the sizes of the extra dimensions while at the smaller scales the system consists of a particle - ghost-like pair given in (48). This effectively amounts to Pauli-Villars-like regularization that is essentially equivalent to Pauli-Villars regularization whenever a single fermion internal line is involved in a Feynman diagram. The situation for higher number of fermion internal lines is nontrivial and needs further study. The indication that the extra dimensional metric reversal symmetry may be relevant to regularization of the divergences of quantum field theory in addition to its relevance to the cosmological constant and zero point energy problems may suggest that this symmetry may be a fundamental symmetry of nature with much deeper implications. However this study should be regarded as an example of the possibility of inducing a Pauli-Villars like regularization through extra dimensions rather than a generic possibility since the realization of this scheme needs non-trivial boundary conditions and extra-dimensional symmetries. In my opinion a more detailed study of these points, and the extension of this study to other fields, such as gauge and scalar fields, and its possible relation with Lee-Wick model [15] should be considered in future. I do not anticipate extreme difficulty in the extension of this scheme to the other fields provided that the discrete symmetries and the anti-periodic boundary conditions employed here for some of the extra dimensions is used in these extensions as well. All these points need separate studies by their own.

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